Weak BAB conjecture for log del Pezzo surfaces and Mori fiber spaces

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Definitions

We work over complex number field.
Let X be a normal projective variety and Δ be an $\mathbb{Q}$-divisor on X with coefficients in $[0,1]$ such that $K_X+\Delta$ is $\mathbb{Q}$-Cartier. We say that (X,Δ) is a *log Fano variety* if $-(K_X+\Delta)$ is ample. In dimension two, it is called log del Pezzo surface. Let $f: Y \to X$ be a log resolution of (X,Δ), write

$$K_Y = f^*(K_X+\Delta)+\sum a_i F_i,$$

where F_i is a prime divisor. For some $\epsilon \in (0,1]$, the pair (X,Δ) is called *ϵ-kawamata log terminal* (ϵ-*klt*, for short) if $a_i > -1+\epsilon$ for all i, or *ϵ-log canonical* (ϵ-*lc*, for short) if $a_i \geqslant -1+\epsilon$ for all i.

Boundedness on log Fano varieties

One of the most interesting conjecture in minimal model theory is the following B-A-B Conjecture due to A. Borisov, L. Borisov and V. Alexeev.

(Birational) BAB conjecture

Fix $0<\epsilon<1$, an integer $n>0$, and consider the set of all n-dimensional ϵ-klt log Fano varieties (X,Δ). The set of underlying varieties $\{X\}$ is (birationally) bounded.

The BAB Conjecture is still open in dimension three and higher. We are mainly interested in the following weak conjecture for anti-canonical volumes which is a consequence of BAB Conjecture.

Weak BAB conjecture

Fix $0<\epsilon<1$, an integer $n>0$, and consider the set of all n-dimensional ϵ-klt log Fano varieties (X,Δ). The volume $\mathrm{Vol}(-(K_X+\Delta))=(-(K_X+\Delta))^n$ is bounded from above by a fixed number $M(n,\epsilon)$ depending only on n and ϵ.

Weak BAB conjecture in dimension two

In dimension two, the conjecture is well-researched in history. We give an optimal bound.

Theorem A

Let (X,Δ) be an ϵ-lc weak log del Pezzo surface. Then the anti-canonical volume $\mathrm{Vol}(-(K_X+\Delta))=(K_X+\Delta)^2$ satisfies

$$(K_X+\Delta)^2 \leqslant \max\left\{9, \lfloor 2/\epsilon\rfloor+4+\frac{4}{\lfloor 2/\epsilon\rfloor}\right\},$$

where $\lfloor\ \rfloor$ means round down.
Moreover, the equality holds if and only if one of the following holds:
(1) $\epsilon>\frac{2}{5}$ and (X,Δ) is $(\mathbb{P}^2,0)$;
(2) $\epsilon\leqslant\frac{1}{2}$ and (X,Δ) is $(\mathbb{F}_n,(1-\frac{2}{n})S_n)$ or $(\mathrm{PC}_n,0)$, where $n=\lfloor 2/\epsilon\rfloor$, $\mathbb{F}_n$ is the n-th Hirzebruch surface, $S_n\subset\mathbb{F}_n$ is the unique curve with negative self-intersection and PC_n is the projective cone over a rational normal curve of degree n.

BAB conjecture for Mori fiber spaces

A variety X is said to be with a *Mori fiber space* structure if X has $\mathbb{Q}$-factorial terminal singularities and there is a morphism $f: X\to S$ such that $-K_X$ is ample on fibers, $\rho(X/S)=1$ and $\dim X>\dim S$. We have the following facts.

Fact

BAB $\Longrightarrow$ BAB for Mfs $\Longrightarrow$ birational BAB
'weak BAB' for Mfs $\Longrightarrow$ weak BAB

So it is very interesting to investigate the boundedness of Mori fiber spaces.

3-fold Mori fiber spaces

There are three possible Mori fiber space structures on a 3-fold log Fano pair (X,Δ).
Case 0: $\dim S=0$
In this case X is a terminal Fano 3-fold with Picard number one. The BAB conjecture is true for this class of varieties by Kawamata. And the optimal bound for the anti canonical volume is given by Prokhorov.
Case 1: $\dim S=1$
In this case S is just $\mathbb{P}^1$ and the general fiber is smooth del Pezzo surface. We can prove BAB conjecture for $\mathbb{P}^2$-bundles over $\mathbb{P}^1$, or more general, $\mathbb{P}^n$-bundles over $\mathbb{P}^1$. The method is to compute discrepancies for some special centers to get some bound.

Theorem B

$X=\mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1}\oplus\mathcal{O}_{\mathbb{P}^1}(a_1)\oplus\cdots\oplus\mathcal{O}_{\mathbb{P}^1}(a_n))$ with $0\le a_1\le\cdots\le a_n$, and (X,Δ) is ϵ-klt log Fano for some Δ. Then $a_n<2/\epsilon$.

Case 2: $\dim S=2$
In this case X is a conic bundle over S. We can prove weak BAB conjecture for the case $S=\mathbb{P}^2$. The idea is that if the volume is big then we can construct non-klt centers and using connected lemma to get special non-klt centers which is very big on the fibers.

Theorem C

Let $X\to\mathbb{P}^2$ be a conic bundle, and (X,Δ) is ϵ-klt log Fano for some Δ. Then $-(K_X+\Delta)^3\le(12/\epsilon)^3$.